1. Si $L\left[f'(t)\right] = \dfrac{s}{(s^2-1)^2}$ y $f(0) = 0,$

   a) determine $L[f(t)]$.

   b) calcule $\int_0^{+\infty} e^{-4t} f(t)dt$

2. Si $L\left[f''(t)\right] = \dfrac{3s^3+s+s^2-1}{(s+1)(s-1)(s^2+1)}$ y $f(0) = 3,$ $f'(0) = 1$,

   determine $L[f(t)]$.

3. Determine $L[5^{at}\cosh(bt)]$, donde a y b son constantes reales no nulas.

4. Si $L[e^{2t}f(t)] = \left[\dfrac{2s-1}{s^2-4s+13}\right]$ y $f(0) = 2$, determine $L\left[e^{-2t}f'(t)\right]$.

5. Si $L^{-1}[F(s)] = e^{at}\cos(bt)$ donde a y b son constantes reales no nulas,

   determine

   a) $L\left[e^{at}f'(t)\right]$

   b) $L^{-1}[F(s-c)]$ donde c es una constante real positiva

6. Sean a y b son constantes reales no nulas.

   Si $L[x'(t)+y'(t)] = -\dfrac{as+2ab+bs}{(s+a)(s+b)}$ y $x(0) = y(0) = 1,$

   determine $L[x(t)+y(t)]$.

7. Sean a y b son constantes reales positivas.

   Si $x'(t) = be^{-at}y(t)$, $L[x(t)] = \dfrac{a^2+b^2}{(s+a)(s^2+b^2)}$ y $x(0) = 0,$

   determine $L[y(t)]$.

8. Si $y''(t)+y'(t) = \cos(t)-\sin(t)$ y $y(0) = 0$, $y'(0) = 1$, determine $L\left[e^{5t}y'(t)\right]$.

9. Si $y'(t) = g(t)$ y $y(0) = 1$, $y'(0) = 0$, $L[g(t)] = G(s)$, determine $L\left[\sinh(t)y''(t)\right]$.

10. Sean a y b son constantes reales positivas.

Si $F(s) = \dfrac{-s}{s^2 + 2as + 2a^2 + 2ab + b^2}$, determine $L^{-1}[F(s+a) + F(s-a)]$.

11. Sea a una constante real positiva.

Determine $L\left[(f(t) + e^{at})^2\right]$ si $L\left[(f(t))^2\right] = \dfrac{s^2 + 2a^2}{s(s^2 + 4a^2)}$ y $f(0) = 1$.

12. Si $L[e^{2t}f(t)] = \dfrac{s+2}{(s+2)^2 + 4}$, calcule $\int_0^{+\infty} e^{-6t} f'(t)dt$.

13. Sea $n \in IN$

Si $L[f(t)] = \dfrac{1}{(s-n)^n} + \dfrac{1}{(s+n)^{2n}}$, determine

a) $f(t)$

b) $\int_0^{+\infty} e^{-3nt}\cosh(nt) f(t)dt$

14. Sea a una constante real no nula.

Si $y''(t) + y'(t) = a(a+1)e^{at}$ tal que $y(1) = e^a$ y $y(2) = e^{2a}$, determine $y(t)$.

Resolución

1. Si $L\left[f'(t)\right] = \dfrac{s}{(s^2-1)^2}$ y $f(0) = 0$,

   a) determine $L[f(t)]$.

   b) calcule $\int_0^{+\infty} e^{-4t} f(t) d$

Solución:

a) $L\left[f'(t)\right] = sF(s) - f(0^+) = sF(s)$

$$L\left[f'(t)\right] = \frac{s}{(s^2-1)^2} \qquad \Rightarrow \qquad sF(s) = \frac{s}{(s^2-1)^2}$$

$$F(s) = \frac{1}{(s^2-1)^2}$$

b) $\displaystyle\int_0^{+\infty} e^{-4t} f(t)dt = \frac{1}{(4^2-1)^2} = \frac{1}{225}$

2. Si $L\left[f''(t)\right] = \dfrac{3s^3+s+s^2-1}{(s+1)(s-1)(s^2+1)}$ y $f(0) = 3$, $f'(0) = 1$,

   determine $L[f(t)]$

Solución:

$$L\left[f''(t)\right] = s^2F(s) - sf(0^+) - f'(0^+) = s^2F(s) - 3s - 1$$

$$L\left[f''(t)\right] = \frac{3s^3+s+s^2-1}{(s+1)(s-1)(s^2+1)} \qquad \Rightarrow \qquad s^2F(s) - 3s - 1 = \frac{3s^3+s+s^2-1}{(s+1)(s-1)(s^2+1)}$$

$$s^2F(s) = 3s + 1 + \frac{3s^3+s+s^2-1}{(s+1)(s-1)(s^2+1)}$$

$$s^2F(s) = \frac{3s^5-2s+s^4-2+3s^3+s^2}{(s+1)(s-1)(s^2+1)}$$

$$F(s) = \frac{3s^5-2s+s^4-2+3s^3+s^2}{s^2(s+1)(s-1)(s^2+1)}$$

3. Determine $L[5^{at}\cosh(bt)]$, donde a y b son constantes reales no nulas.

Solución:

$$L[5^{at}\cosh(bt)] = L\left[5^{at}\left(\frac{e^{bt}+e^{-bt}}{2}\right)\right]$$

$$= \frac{1}{2}L[5^{at}(e^{bt}+e^{-bt})] = \frac{1}{2}L[5^{at}e^{bt}+5^{at}e^{-bt}]$$

$$= \frac{1}{2}L[e^{at\ln(5)}e^{bt}+e^{at\ln(5)}e^{-bt}]$$

$$= \frac{1}{2}L[e^{at\ln(5)+bt}+e^{at\ln(5)-bt}]$$

$$= \frac{1}{2}L[e^{(a\ln(5)+b)t}+e^{(a\ln(5)-b)t}]$$

$$= \frac{1}{2}\left(\frac{1}{s-(a\ln(5)+b)}+\frac{1}{s-(a\ln(5)-b)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a\ln(5)-b}+\frac{1}{s-a\ln(5)+b}\right)$$

4. Si $L[e^{2t}f(t)] = \left[\frac{2s-1}{s^2-4s+13}\right]$ y $f(0) = 2$, determine $L\left[e^{-2t}f'(t)\right]$.

Solución:

$$L[e^{2t}f(t)] = \left[\frac{2s-1}{s^2-4s+13}\right] = \left[\frac{2(s-2+2)-1}{(s-2)^2-4+13}\right]$$

$$L[e^{2t}f(t)] = \left[\frac{2(s-2)+3}{(s-2)^2+9}\right]$$

$$L[f(t)] = \left[\frac{2s+3}{s^2+9}\right]$$

$$L\left[f'(t)\right] = sF(s) - f(0^+) = sF(s) - 2$$

$$L\left[f'(t)\right] = s \bullet \frac{2s+3}{s^2+9} - 2 = \frac{3s-18}{s^2+9}$$

$$L\left[e^{-2t}f'(t)\right] = \frac{3(s+2)-18}{(s+2)^2+9} = \frac{3s-12}{s^2+4s+13}$$

5. Si $L^{-1}[F(s)] = e^{at}\cos(bt)$ donde a y b son constantes reales no nulas,

determine

a) $L\left[e^{at}f'(t)\right]$

b) $L^{-1}[F(s-c)]$ donde c es una constante real positiva

Solución:

a) $L[\cos(bt)] = \dfrac{s}{s^2+b^2}$

$$F(s) = \frac{s-a}{(s-a)^2+b^2}$$

$$L\left[f'(t)\right] = sF(s) - f(0^+) = s \bullet \frac{s-a}{(s-a)^2+b^2} - 1 = \frac{s^2-as}{(s-a)^2+b^2} - 1$$

$$L\left[f'(t)\right] = \frac{as-a^2-b^2}{s^2-2as+a^2+b^2}$$

$$L\left[e^{at}f'(t)\right] = \frac{a(s-a)-a^2-b^2}{(s-a)^2-2a(s-a)+a^2+b^2} = \frac{as-2a^2-b^2}{s^2-4as+4a^2+b^2}$$

b) $L^{-1}[F(s-c)] = e^{ct}e^{at}\cos(bt) = e^{(a+c)t}\cos(bt)$

6. Sean a y b son constantes reales no nulas.

Si $L[x'(t)+y'(t)] = -\dfrac{as+2ab+bs}{(s+a)(s+b)}$ y $x(0) = y(0) = 1$,

determine $L[x(t)+y(t)]$.

Solución:

$$L[x'(t)+y'(t)] = L[x'(t)] + L[y'(t)] = sX(s) - x(0^+) + sY(s) - y(0^+)$$

$$L[x'(t)+y'(t)] = sX(s) - 1 + sY(s) - 1 = s(X(s)+Y(s)) - 2$$

$$-\frac{as+2ab+bs}{(s+a)(s+b)} = s(X(s)+Y(s)) - 2$$

$$X(s)+Y(s) = \frac{1}{s}\left(-\frac{as+2ab+bs}{(s+a)(s+b)} + 2\right) = \frac{a+b+2s}{(s+a)(s+b)}$$

7. Sean a y b son constantes reales positivas.

Si $x'(t) = be^{-at}y(t)$, $L[x(t)] = \dfrac{a^2+b^2}{(s+a)(s^2+b^2)}$ y $x(0) = 0$,

determine $L[y(t)]$.

Solución:

$$L\left[x'(t)\right] = sX(s) - x(0^+) = sX(s)$$

$$L\left[x'(t)\right] = L[be^{-at}y(t)] = bY(s+a)$$

$$sX(s) = bY(s+a)$$

$$Y(s+a) = \frac{sX(s)}{b} = \frac{(a^2+b^2)s}{b(s+a)(s^2+b^2)}$$

$$Y(s) = \frac{(a^2+b^2)(s-a)}{bs\left((s-a)^2+b^2\right)}$$

8. Si $y''(t) + y'(t) = \cos(t) - \sin(t)$ y $y(0) = 0$, $y'(0) = 1$, determine $L\left[e^{5t}y'(t)\right]$.

Solución:

$$y''(t) + y'(t) = \cos(t) - \sin(t) \qquad / \ L$$

$$L\left[y''(t) + y'(t)\right] = L[\cos(t) - \sin(t)]$$

$$s^2Y(s) - sy(0^+) - y'(0^+) + sY(s) - y(0^+) = \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$s^2Y(s) - 1 + sY(s) = \frac{s-1}{s^2+1}$$

$$(s^2+s)Y(s) = \frac{s-1}{s^2+1} + 1$$

$$(s^2+s)Y(s) = \frac{s^2+s}{s^2+1}$$

$$Y(s) = \frac{1}{s^2+1}$$

$$L\left[y'(t)\right] = sY(s) - y(0^+) = \frac{s}{s^2+1}$$

$$L\left[e^{5t}y'(t)\right] = \frac{s-5}{(s-5)^2+1} = \frac{s-5}{s^2-10s+26}$$

9. Si $y'(t) = g(t)$ y $y(0) = 1$, $y'(0) = 0$, $L[g(t)] = G(s)$, determine $L\left[\sinh(t)y''(t)\right]$.

Solución:

$y'(t) = g(t) \quad / \quad L$

$L\left[y'(t)\right] = L[g(t)]$

$sY(s) - y(0^+) = G(s)$

$sY(s) - 1 = G(s)$

$sY(s) = G(s) + 1$

$Y(s) = \dfrac{G(s)+1}{s}$

$L\left[y''(t)\right] = s^2Y(s) - s$

$L\left[y''(t)\right] = s^2 \bullet \dfrac{G(s)+1}{s} - s$

$L\left[y''(t)\right] = s(G(s)+1) - s$

$L\left[y''(t)\right] = sG(s)$

$L\left[\sinh(t)y''(t)\right] = L\left[\left(\dfrac{e^t - e^{-t}}{2}\right)y''(t)\right] = \dfrac{1}{2}\left(L\left[e^t y''(t)\right] - L\left[e^{-t}y''(t)\right]\right)$

$L\left[\sinh(t)y''(t)\right] = \dfrac{1}{2}((s-1)G(s-1) - (s+1)G(s+1))$

10. Sean a y b son constantes reales positivas.

Si $F(s) = \dfrac{-s}{s^2 + 2as + 2a^2 + 2ab + b^2}$, determine $L^{-1}[F(s+a) + F(s-a)]$.

Solución:

$$F(s) = \frac{-s}{s^2 + 2as + 2a^2 + 2ab + b^2} = \frac{-s}{(s+a)^2 - a^2 + 2a^2 + 2ab + b^2}$$

$$F(s) = \frac{-s}{(s+a)^2 + a^2 + 2ab + b^2} = \frac{-(s+a-a)}{(s+a)^2 + a^2 + 2ab + b^2}$$

$$F(s) = \frac{-(s+a) + a}{(s+a)^2 + (a+b)^2} = \frac{-(s+a)}{(s+a)^2 + (a+b)^2} + \frac{a}{(s+a)^2 + (a+b)^2}$$

$$F(s) = -\frac{(s+a)}{(s+a)^2 + (a+b)^2} + \frac{a}{a+b} \bullet \frac{a+b}{(s+a)^2 + (a+b)^2}$$

$$f(t) = e^{-at}\left(\cos((a+b)t) + \frac{a\sin((a+b)t)}{a+b}\right)$$

$$L^{-1}[F(s+a) + F(s-a)] = e^{-at} \bullet f(t) + e^{at} \bullet f(t) = (e^{-at} + e^{at})f(t)$$

$$L^{-1}[F(s+a) + F(s-a)] = (e^{-at} + e^{at}) \bullet e^{-at}\left(\cos((a+b)t) + \frac{a\sin((a+b)t)}{a+b}\right)$$

$$L^{-1}[F(s+a) + F(s-a)] = (e^{-2at} + 1)\left(\cos((a+b)t) + \frac{a\sin((a+b)t)}{a+b}\right)$$

11. Sea a una constante real positiva.

Determine $L\left[(f(t) + e^{at})^2\right]$ si $L\left[(f(t))^2\right] = \dfrac{s^2 + 2a^2}{s(s^2 + 4a^2)}$ y $f(0) = 1$.

Solución:

$$L\left[(f(t) + e^{at})^2\right] = L\left[(f(t))^2 + 2f(t)e^{at} + e^{2at}\right]$$

$$= L\left[(f(t))^2\right] + 2L[f(t)e^{at}] + L[e^{2at}]$$

$$= \frac{s^2 + 2a^2}{s(s^2 + 4a^2)} + 2L[e^{at}f(t)] + \frac{1}{s - 2a}$$

$$L\left[(f(t))^2\right] = \frac{s^2 + 2a^2}{s(s^2 + 4a^2)} = \frac{1}{2s} + \frac{1}{2} \bullet \frac{s}{s^2 + 4a^2}$$

$$(f(t))^2 = \frac{1}{2} + \frac{1}{2}\cos(2at) = \cos^2(at)$$

$f(t) = \cos(at)$

$L[f(t)] = \dfrac{s}{s^2 + a^2}$

$L[e^{at}f(t)] = \dfrac{s-a}{(s-a)^2 + a^2} = \dfrac{s-a}{s^2 - 2as + 2a^2}$

$L\left[(f(t) + e^{at})^2 \right] = \dfrac{s^2 + 2a^2}{s(s^2 + 4a^2)} + \dfrac{2(s-a)}{s^2 - 2as + 2a^2} + \dfrac{1}{s - 2a}$

12. Si $L[e^{2t}f(t)] = \dfrac{s+2}{(s+2)^2 + 4}$, calcule $\int_0^{+\infty} e^{-6t} f'(t)dt$.

$e^{2t}f(t) = L^{-1}\left[\dfrac{s+2}{(s+2)^2 + 4} \right] = e^{-2t}\cos(2t)$

$f(t) = e^{-4t}\cos(2t)$

$f(0) = 1$

$L[f(t)] = \dfrac{s+4}{(s+4)^2 + 4}$

$L\left[f'(t)\right] = sF(s) - f(0^+) = \dfrac{s(s+4)}{(s+4)^2 + 4} - 1$

$\int_0^{+\infty} e^{-6t} f'(t)dt = \dfrac{6 \bullet (6+4)}{(6+4)^2 + 4} - 1 = -\dfrac{11}{26}$

13. Sea $n \in IN$

Si $L[f(t)] = \dfrac{1}{(s-n)^n} + \dfrac{1}{(s+n)^{2n}}$, determine

a) $f(t)$

b) $\int_0^{+\infty} e^{-3nt}\cosh(nt)f(t)dt$

Solución:

a) $L[f(t)] = \dfrac{1}{(n-1)!} \bullet \dfrac{(n-1)!}{(s-n)^n} + \dfrac{1}{(2n-1)!} \bullet \dfrac{(2n-1)!}{(s+n)^{2n}}$

$$f(t) = \frac{e^{nt}t^{n-1}}{(n-1)!} + \frac{e^{-nt}t^{2n-1}}{(2n-1)!}$$

b) $L[\cosh(nt)f(t)] = \dfrac{1}{2}L[(e^{nt}+e^{-nt})f(t)]$

$$L[\cosh(nt)f(t)] = \frac{1}{2}(F(s-n)+F(s+n))$$

$$= \frac{1}{(s-2n)^n} + \frac{1}{s^{2n}} + \frac{1}{s^n} + \frac{1}{(s+2n)^{2n}}$$

$$\int_0^{+\infty} e^{-3nt}\cosh(nt)f(t)dt = \frac{1}{(3n-2n)^n} + \frac{1}{(3n)^{2n}} + \frac{1}{(3n)^n} + \frac{1}{(3n+2n)^{2n}}$$

$$\int_0^{+\infty} e^{-3nt}\cosh(nt)f(t)dt = \frac{1}{n^n} + \frac{1}{(3n)^{2n}} + \frac{1}{(3n)^n} + \frac{1}{(5n)^{2n}}$$

14. Sea a una constante real no nula.

Si $y''(t) + y'(t) = a(a+1)e^{at}$ tal que $y(1) = e^{a}$ y $y(2) = e^{2a}$, determine $y(t)$.

Solución:

$$y'' + y' = a(a+1)e^{at} \qquad / \; L$$

$$s^2Y(s) - sy(0^+) - y'(0^+) + sY(s) - y(0^+) = \frac{a(a+1)}{s-a}$$

$$s^2Y(s) + sY(s) = sy(0^+) + y'(0^+) + y(0^+) + \frac{a(a+1)}{s-a}$$

$$(s^2 + s)Y(s) = sy(0^+) + y'(0^+) + y(0^+) + \frac{a(a+1)}{s-a}$$

$$s(s+1)Y(s) = sy(0^+) + y'(0^+) + y(0^+) + \frac{a(a+1)}{s-a}$$

$$Y(s) = \frac{sy(0^+)}{s(s+1)} + \frac{y'(0^+) + y(0^+)}{s(s+1)} + \frac{a(a+1)}{s(s+1)(s-a)}$$

$$Y(s) = \frac{y(0^+)}{s+1} + \frac{y'(0^+) + y(0^+)}{s(s+1)} + \frac{a(a+1)}{s(s+1)(s-a)}$$

Expandiendo en fracciones parciales:

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\frac{a(a+1)}{s(s+1)(s-a)} = -\frac{a+1}{s} + \frac{a}{s+1} + \frac{1}{s-a}$$

$$Y(s) = \frac{y(0^+)}{s+1} + \left(y'(0^+) + y(0^+)\right)\left(\frac{1}{s} - \frac{1}{s+1}\right) - \frac{a+1}{s} + \frac{a}{s+1} + \frac{1}{s-a}$$

$$Y(s) = \frac{y(0^+)}{s+1} + \frac{y'(0^+)}{s} - \frac{y'(0^+)}{s+1} + \frac{y(0^+)}{s} - \frac{y(0^+)}{s+1} - \frac{a+1}{s} + \frac{a}{s+1} + \frac{1}{s-a}$$

$$Y(s) = \frac{y'(0^+)}{s} - \frac{y'(0^+)}{s+1} + \frac{y(0^+)}{s} - \frac{a+1}{s} + \frac{a}{s+1} + \frac{1}{s-a}$$

$$y(t) = y'(0^+) - y'(0^+)e^{-t} + y(0^+) - (a+1) + ae^{-t} + e^{at}$$

$$y(1) = e^{a} \quad \Rightarrow \quad e^{a} = y'(0^+) - y'(0^+)e^{-a} + y(0^+) - (a+1) + ae^{-a} + e^{a}$$

$$y'(0^+)(1 - e^{-a}) + y(0^+) = a + 1 - ae^{-a}$$

$$y(2) = e^{2a} \quad \Rightarrow \quad e^{2a} = y'(0^+) - y'(0^+)e^{-2a} + y(0^+) - (a+1) + ae^{-2a} + e^{2a}$$

$$y'(0^+)(1 - e^{-2a}) + y(0^+) = a + 1 - ae^{-2a}$$

Resolviendo el sistema de ecuaciones para determinar $y(0^+)$ y $y'(0^+)$ se tiene:

$$y'(0^+)(1-e^{-a})+y(0^+) = a+1-ae^{-a}$$
$$y'(0^+)(1-e^{-2a})+y(0^+) = a+1-ae^{-2a}$$

Restando:

$$y'(0^+)(1-e^{-a}-1+e^{-2a}) = a+1-ae^{-a}-(a+1)+ae^{-2a}$$

$$y'(0^+)(-e^{-a}+e^{-2a}) = -ae^{-a}+ae^{-2a}$$

$$y'(0^+) = \frac{-ae^{-a}+ae^{-2a}}{-e^{-a}+e^{-2a}} = a$$

Reemplazando:

$$y'(0^+)(1-e^{-a})+y(0^+) = (a+1)-ae^{-a}$$

$$a(1-e^{-a})+y(0^+) = a+1-ae^{-a}$$

$$y(0^+) = a+1-ae^{-a}-a(1-e^{-a}) = 1$$

Reemplazando $y(0^+) = 1$, $y'(0^+) = a$ en la expresión para $y(t)$:

$$y(t) = y'(0^+)-y'(0^+)e^{-t}+y(0^+)-(a+1)+ae^{-t}+e^{at}$$

$$y(t) = a-ae^{-t}+1-(a+1)+ae^{-t}+e^{at} = e^{at}$$

Luego, $y(t) = e^{at}$