Sergio Yansen Núñez

Ejercicios de Transformada de Laplace

1. Determine la transformada de Laplace de cada una de las siguientes funciones:

a) $f(t) = e^{5t}(3t-2)^2$

b) $f(t) = e^{-4t} sen(3t) cos(3t)$

2. Usando la definición de transformada de Laplace, determine las siguientes integrales impropias:

a) $\int_0^{+\infty} (t^3+1)e^{-2t} dt$

b) $\int_0^{+\infty} e^{-2t} cos^2(t) dt$

3. a) Determine $\mathcal{L}\left[(e^{2t}-t)^2\right]$

b) Usando a), determine $\int_0^{+\infty} e^{-5t}(e^{2t}-t)^2 dt$

4. Sea $F(s) = \mathcal{L}\left[f(t)\right] = \frac{s-1}{s^2-4s+13}$. Determine:

a) $f(t)$

b) $\mathcal{L}\left[e^{-2t} \cdot f(t)\right]$

c) $\mathcal{L}\left[f'(t)\right]$

d) $\mathcal{L}\left[f(t) \cdot \mu(t-2)\right]$

e) $\mathcal{L}\left[e^{-2t} \cdot f(t) \cdot \mu(t-2)\right]$

f) $\mathcal{L}^{-1}\left[e^{-2s} \cdot F(s)\right]$

g) $\int_0^{+\infty} e^{-2t} f(t) dt$

5. El movimiento de un sistema de oscilación mecánica está gobernado por la ecuación diferencial $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$.
donde m es la masa, c es la constante de amortiguamiento, k es la constante del resorte, $f(t)$ es una fuerza externa.
La constante de un resorte de acero de 2 pie se mide colgando una masa que pesa 16 lb del resorte y observando que el resorte se estira $\frac{1}{4}$ pie . Ahora se cuelga del resorte una masa que pesa 8 lb . La masa se jala hacia abajo $\frac{1}{4}$ pie y se deja libre con una velocidad dirigida hacia abajo de 1 pie/seg . Sea $x(t)$ el desplazamiento para $t \geq 0$. Determine $\mathcal{L}\left[x(t)\right]$.

6. Resuelva la siguiente ecuación diferencial para $t \geq 0$

$$y'' - 4y' + 4y = f(t) \quad \text{donde} \quad f(t) = \begin{cases} 4 & \text{si } 0 \leq t < 2 \\ 2t & \text{si } t \geq 2 \end{cases}$$

$y(0) = 0$, $y'(0) = 0$

7. Mediante la transformada de Laplace resuelva el siguiente sistema de ecuaciones diferenciales:

$$\frac{dx}{dt} + x + y = 2cos(2t)$$

$$\frac{dy}{dt} + x - y = -2cos(2t) + 2sen(2t)$$

sujeto a $x(0) = 0$; $y(0) = 0$

8. Si $\mathcal{L}[e^t f(t)] = \frac{s^2-2s-15}{(s^2-2s+17)^2}$ y $f(0) = 0$, determine:

a) $\mathcal{L}[f(t)]$

b) $\mathcal{L}[e^{2t} f(t)]$

c) $\int_0^{+\infty} e^{-4t} f(t)dt$

d) $\mathcal{L}[te^{2t} f(t)]$

e) $\mathcal{L}[f'(t)]$

f) $\mathcal{L}\left[\int_0^t e^u f(u)du\right]$

g) $\mathcal{L}^{-1}\left[\frac{e^{-5s}((s-2)^2-2(s-2)-15)}{((s-2)^2-2(s-2)+17)^2}\right]$

h) $\mathcal{L}[e^{t-2} f(t-2)\mu(t-2)]$

9. Sea $f(t) = ((t-1)e^{t-1}\int_0^{t-1} u^8 e^{4u} du)\mu(t-1)$. Determine $\mathcal{L}[f(t)]$

10. La ecuación diferencial para la corriente $i(t)$ en un circuito en serie que contiene un inductor y un resistor es $L\frac{di}{dt} + Ri = E(t)$

donde $E(t)$ es la tensión aplicada. Use la transformada de Laplace para determinar la corriente $i(t)$ cuando $i(0) = 0$. Considere $L = 1\,H$, $R = 10\,\Omega$, y

$$E(t) = \begin{cases} 0 & , t < 0 \\ sen(t) & , 0 \leq t < \frac{3\pi}{2} \\ 0 & , t \geq \frac{3\pi}{2} \end{cases}$$

11. Sea $f(t) = sen(4t) + 4tcos(4t)$. Si $\mathcal{L}[f(t)] = \frac{8s^2}{(s^2+16)^2}$, determine :

a) $\mathcal{L}\left[\frac{f(t)}{e^{3t+1}}\right]$

b) $\mathcal{L}[t\int_0^t f(u)du]$

c) $\int_0^{+\infty} e^{-3t} f(t)dt$

d) $\mathcal{L}^{-1}\left[\frac{e^{-2s}(s-2)^2}{(s^2-4s+20)^2}\right]$

12. Resuelva el siguiente sistema de ecuaciones diferenciales:

$$\frac{dx}{dt} = 1 + 3x - y$$

$$\frac{dy}{dt} = 2x$$

sujeto a $x(0) = 1$, $y(0) = 2$.

13. Mediante la transformada de Laplace resuelva el siguiente sistema de ecuaciones diferenciales:

$$\frac{dx}{dt} + x + y = 2cos(2t)$$

$$\frac{dy}{dt} + x - y = -2cos(2t) + 2sen(2t)$$

$$x(0) = 0 \quad ; \quad y(0) = 0$$

14. Resuelva la ecuación diferencial $y'' + y = \delta(t - 2\pi)$ sujeta a $y(0) = 1,\ y'(0) = 0$.

15. Sea $f(t) = sen(3t) + 3tcos(3t)$. Si $\mathcal{L}[f(t)] = \frac{6s^2}{(s^2+9)^2}$, determine:

a) $\mathcal{L}\left[\frac{f(t)}{e^{2t+1}}\right]$

b) $\int_0^{+\infty} e^{-2t} f(t)dt$

c) $\mathcal{L}^{-1}\left[\frac{e^{-3s}(s-3)^2}{(s^2-6s+18)^2}\right]$

16. Si $\mathcal{L}[f(t)] = \frac{2s^2-s-2}{s(s+2)(s-2)}$, determine

a) $\mathcal{L}^{-1}[F(s)]$

b) $\mathcal{L}[e^{t-1} f(t-1)\mu(t-1)]$

17. Resuelva mediante tansformada de Laplace la siguiente ecuación diferencial sujeta a las condiciones indicadas.

$$y'' - y' = e^t \qquad y(0) = 2 \quad , \quad y'(0) = 2$$

18. Usando la transformada de Laplace, determine una solución polinomial de grado 1 de la ecuación diferencial

$$ty''(t) + (1-t)y'(t) + y(t) = 0$$

19. Determine $\mathcal{L}[g(t)]$ si $g(t) = \int_1^t e^x (x - [\![x]\!]) dx$

20. Sean

$$f(t) = \begin{cases} 0 & \text{si } 0 \le t < \pi \\ -\,sen(t) & \text{si } t \ge \pi \end{cases}$$

$$g(t) = \begin{cases} t & \text{si } 0 \le t < \pi \\ 0 & \text{si } t \ge \pi \end{cases}$$

a) Exprese f y g en término de funciones escalones (de Heaviside)

b) Determine $(f*g)(t)$

Resolución

1. Determine la transformada de Laplace de cada una de las siguientes funciones:

a) $f(t) = e^{5t}(3t-2)^2$

Solución:

$(3t-2)^2 = 9t^2 - 12t + 4$

$\mathcal{L}\big[9t^2 - 12t + 4\big] = 9 \cdot \frac{2}{s^3} - 12 \cdot \frac{1}{s^2} + \frac{4}{s}$

$\mathcal{L}\big[e^{5t}(3t-2)^2\big] = \frac{18}{(s-5)^3} - \frac{12}{(s-5)^2} + \frac{4}{s-5}$

b) $f(t) = e^{-4t} sen(3t) cos(3t)$

Solución:

$sen(3t)cos(3t) = \frac{sen(6t)}{2}$

$\mathcal{L}\big[\frac{sen(6t)}{2}\big] = \frac{1}{2} \cdot \frac{6}{s^2+36}$

$\mathcal{L}\big[e^{-4t} \cdot \frac{sen(6t)}{2}\big] = \frac{1}{2} \cdot \frac{6}{(s+4)^2+36} = \frac{3}{(s+4)^2+36}$

2. Usando la definición de transformada de Laplace, determine las siguientes integrales impropias:

a) $\int_0^{+\infty} (t^3+1)e^{-2t}dt$

Solución:
$\mathcal{L}\big[t^3+1\big] = \frac{6}{s^4} + \frac{1}{s}$

$\int_0^{+\infty} (t^3+1)e^{-2t}dt \quad = \left(\frac{6}{s^4} + \frac{1}{s}\right)\Big|_{s=2} = \frac{6}{24} + \frac{1}{2} = \frac{7}{8}$

b) $\int_0^{+\infty} e^{-2t} cos^2(t) dt$

Solución:

$\mathcal{L}\big[cos^2(t)\big] = \mathcal{L}\big[\frac{1+cos(2t)}{2}\big] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$

$\int_0^{+\infty} e^{-2t} cos^2(t) dt = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)\Big|_{s=2} = \frac{1}{2}\left(\frac{1}{2} + \frac{2}{4+4}\right) = \frac{3}{8}$

3. a) Determine $\mathcal{L}\left[(e^{2t}-t)^2\right]$

Solución:

$(e^{2t}-t)^2 = e^{4t} - 2te^{2t} + t^2$

$\mathcal{L}\left[(e^{2t}-t)^2\right] = \frac{1}{s-4} - 2\cdot\frac{1}{(s-2)^2} + \frac{2}{s^3}$

b) Usando a), determine $\int_0^{+\infty} e^{-5t}(e^{2t}-t)^2 dt$

Solución:

$$\int_0^{+\infty} e^{-5t}(e^{2t}-t)^2 dt = \left(\frac{1}{s-4} - 2\cdot\frac{1}{(s-2)^2} + \frac{2}{s^3}\right)\Big|_{s=5}$$

$$= \frac{1}{5-4} - 2\cdot\frac{1}{(5-2)^2} + \frac{2}{5^3}$$

$$= \frac{893}{1125}$$

4.

a) $f(t)$

Solución:

$\frac{s-1}{s^2-4s+13} = \frac{s-2}{(s-2)^2+9} + \frac{1}{3}\cdot\frac{3}{(s-2)^2+9}$

$f(t) = e^{2t}(cos(3t) + \frac{1}{3}sen(3t))$

b) $\mathcal{L}\left[e^{-2t}\cdot f(t)\right]$

Solución:

$\mathcal{L}\left[e^{-2t}\cdot f(t)\right] = \frac{s+1}{s^2+9}$

c) $\mathcal{L}\left[f'(t)\right]$

Solución:

$\mathcal{L}\left[f'(t)\right] = sF(s) - f(0) = \frac{s(s-1)}{s^2-4s+13} - 1$

d) $\mathcal{L}\left[f(t)\cdot\mu(t-2)\right]$

Solución:

$\mathcal{L}\left[f(t)\cdot\mu(t-2)\right] = e^{-2s}\mathcal{L}\left[f(t+2)\right]$

$f(t+2) = e^{2(t+2)}(cos(3(t+2)) + \frac{1}{3}sen(3(t+2)))$

$= e^4 e^{2t}(cos(3t)(cos(6) + \frac{sen(6)}{3}) + sen(3t)(\frac{cos(6)}{3} - sen(6))$

$\mathcal{L}\left[f(t)\cdot\mu(t-2)\right] = e^{-2s}e^4\left((cos(6) + \frac{sen(6)}{3})\cdot\frac{s-2}{(s-2)^2+9} + (\frac{cos(6)}{3} - sen(6))\cdot\frac{3}{(s-2)^2+9}\right)$

e) $\mathcal{L}\left[e^{-2t}\cdot f(t)\cdot\mu(t-2)\right]$

Solución:

$\mathcal{L}\left[e^{-2t}\cdot f(t)\cdot\mu(t-2)\right]$

$= e^{-2(s+2)}e^4\left((cos(6) + \frac{sen(6)}{3})\cdot\frac{s}{s^2+9} + (\frac{cos(6)}{3} - sen(6))\cdot\frac{3}{s^2+9}\right)$

$= e^{-2s}\left((cos(6) + \frac{sen(6)}{3})\cdot\frac{s}{s^2+9} + (\frac{cos(6)}{3} - sen(6))\cdot\frac{3}{s^2+9}\right)$

f) $\mathcal{L}^{-1}\left[e^{-2s}\cdot F(s)\right]$

Solución:

$\mathcal{L}^{-1}\left[e^{-2s}\cdot F(s)\right] = e^{2(t-2)}(cos(3(t-2)) + \frac{1}{3}sen(3(t-2)))\cdot\mu(t-2)$

g) $\int_0^{+\infty} e^{-2t}f(t)dt$

Solución:

$\int_0^{+\infty} e^{-2t}f(t)d = F(2) = \frac{1}{9}$

5. $mg = k\triangle l$

$16 = k\frac{1}{4}$

$k = 64\ lb/pie$

Sea $x(t)$ el desplazamiento del cuerpo en el tiempo t.

$mg = 8$, entonces $m = \frac{1}{4}$ slug

$mx'' = -kx$

$\frac{1}{4}x'' = -64x$

$x'' + 256x = 0$

$x(0) = \frac{1}{4} \qquad x'(0) = 1$

$x'' + 256x = 0\ /\mathcal{L}$

$\mathcal{L}[x''] + 256\mathcal{L}[x] = 0$

$s^2X(s) - sx(0) - x'(0) + 256X(s) = 0$

$$s^2X(s) - s \cdot \frac{1}{4} - 1 + 256X(s) = 0$$

$$X(s) = \frac{s \cdot \frac{1}{4} + 1}{s^2 + 256}$$

$$\therefore\ \mathcal{L}\big[x(t)\big] = \frac{s+4}{4(s^2+256)}$$

6. $$f(t) = 4(\mu(t) - \mu(t-2)) + 2t\mu(t-2)$$
$$= 4\mu(t) + (-4 + 2t)\mu(t-2)$$

$$\mathcal{L}\big[f(t)\big] = \frac{4}{s} + e^{-2s}\mathcal{L}\big[-4 + 2(t+2)\big]$$

$$= \frac{4}{s} + e^{-2s}\mathcal{L}\big[2t\big]$$

$$= \frac{4}{s} + e^{-2s} \cdot \frac{2}{s^2}$$

$$= \frac{4}{s} + \frac{2e^{-2s}}{s^2}$$

$$y'' - 4y' + 4y = f(t) \qquad / \mathcal{L}$$

$$s^2Y(s) - y(0^+)s - y'(0^+) - 4(sY(s) - y(0^+)) + 4Y(s) = F(s)$$

$$(s^2 - 4s + 4)Y(s) = \frac{4}{s} + \frac{2e^{-2s}}{s^2}$$

$$(s-2)^2Y(s) = \frac{4}{s} + \frac{2e^{-2s}}{s^2}$$

$$Y(s) = \frac{4}{s(s-2)^2} + \frac{2e^{-2s}}{s^2(s-2)^2}$$

$$Y(s) = 2\left(\frac{2}{s(s-2)^2} + \frac{e^{-2s}}{s^2(s-2)^2}\right)$$

$$y(t) = 2\left(\mathcal{L}^{-1}\left[\frac{2}{s(s-2)^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2(s-2)^2}\right]\right.$$

Descomposión en fracciones parciales de $\frac{2}{s(s-2)^2}$

$$\frac{2}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

Los valores para A, B y C son: $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = 1$

Por tanto, $\mathcal{L}^{-1}\left[\frac{2}{s(s-2)^2}\right] = \frac{1}{2} - \frac{1}{2}e^{2t} + te^{2t}$

Descomposión en fracciones parciales de $\frac{1}{s^2(s-2)^2}$

$$\frac{1}{s^2(s-2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

Los valores para A, B, C y D son: $A = \frac{1}{4}$, $B = \frac{1}{4}$, $C = -\frac{1}{4}$, $D = \frac{1}{4}$

Por tanto, $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)^2}\right] = \frac{1}{4} + \frac{1}{4}t - \frac{1}{4}e^{2t} + \frac{1}{4}te^{2t}$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2(s-2)^2}\right] = \left(\frac{1}{4} + \frac{1}{4}(t-2) - \frac{1}{4}e^{2(t-2)} + \frac{1}{4}(t-2)e^{2(t-2)}\right)\mu(t-2)$$

Luego,

$$y(t) = \quad 2\left(\frac{1}{2} - \frac{1}{2}e^{2t} + te^{2t} + \left(\frac{1}{4} + \frac{1}{4}(t-2) - \frac{1}{4}e^{2(t-2)} + \frac{1}{4}(t-2)e^{2(t-2)}\right)\mu(t-2)\right)$$

7. $\frac{dx}{dt} + x + y = 2cos(2t)$

$$\frac{dy}{dt} + x - y = -2cos(2t) + 2sen(2t)$$

Aplicando Transformada de Laplace a ambas ecuaciones, se obtiene:

$$sX(s) - x(0^+) + X(s) + Y(s) = \frac{2s}{s^2+4}$$

$$sY(s) - y(0^+) + X(s) - Y(s) = -\frac{2s}{s^2+4} + \frac{4}{s^2+4}$$

Ordenando las ecuaciones, se obtiene:

$$(s+1)X(s) + Y(s) = \frac{2s}{s^2+4} \qquad / \cdot (-1)$$

$$X(s) + (s-1)Y(s) = \frac{-2s+4}{s^2+4} \qquad / \cdot (s+1)$$

$$-(s+1)X(s) - Y(s) = -\frac{2s}{s^2+4}$$

$$(s+1)X(s) + (s^2-1)Y(s) = \frac{(-2s+4)(s+1)}{s^2+4}$$

Sumando estas dos últimas ecuaciones:

$$(S^2-2)Y(s) = \frac{(-2s+4)(s+1)}{s^2+4} - \frac{2s}{s^2+4}$$

$(S^2-2)Y(s)=\frac{-2(s^2-2)}{s^2+4}$

$Y(s)=-\frac{2}{s^2+4}$

Aplicando Tansformada de Laplace inversa se obtiene:

$y(t)=-sen(2t)$

De la ecuación $\frac{dy}{dt}+x-y=-2cos(2t)+2sen(2t)$ se obtiene:

$x=y-\frac{dy}{dt}-2cos(2t)+2sen(2t)$

$x=-sen(2t)+2cos(2t)-2cos(2t)+2sen(2t)$

$x=sen(2t)$

Luego, $x(t)=sen(2t)$, $y(t)=-sen(2t)$

8.

a) $\mathcal{L}\left[e^t f(t)\right]=\frac{s^2-2s-15}{(s^2-2s+17)^2}=\frac{(s-1)^2-16}{((s-1)^2+16)^2}$

$\mathcal{L}\left[f(t)\right]=\frac{s^2-16}{(s^2+16)^2}$

b) $\mathcal{L}\left[e^{2t} f(t)\right]=\frac{(s-2)^2-16}{((s-2)^2+16)^2}$

c) $\int_0^\infty e^{-4t}f(t)dt=F(4)=\left.\frac{s^2-16}{(s^2+16)^2}\right|_{s=4}=0$

d) $\mathcal{L}\left[tf(t)\right]=-1\cdot\frac{d}{ds}\left(\frac{s^2-16}{(s^2+16)^2}\right)$

$=-\frac{2s(s^2+16)^2+(s^2-16)\cdot 2(s^2+16)\cdot 2s)}{(s^2+16)^4}$

$=-\frac{2s(s^2+16)+4s(s^2-16)}{(s^2+16)^3}$

$\mathcal{L}\left[te^{2t}f(t)\right]=-\frac{2(s-2)((s-2)^2+16)+4(s-2)((s-2)^2-16)}{((s-2)^2+16)^3}$

e) $\mathcal{L}\left[f'(t)\right]\quad=sF(s)-f(0^+)$

$=s\cdot\frac{s^2-16}{(s^2+16)^2}-0=\frac{s(s^2-16)}{(s^2+16)^2}$

f) $\mathcal{L}\left[\int_0^t e^u f(u)du\right] = \frac{1}{s}\mathcal{L}\left[e^t f(t)\right] = \frac{1}{s} \cdot \frac{s^2-2s-15}{(s^2-2s+17)^2}$

g) $\mathcal{L}^{-1}\left[\frac{s^2-2s-15}{(s^2-2s+17)^2}\right] = e^t f(t)$

$\mathcal{L}^{-1}\left[\frac{(s-2)^2-2(s-2)-15}{((s-2)^2-2(s-2)+17)^2}\right] = e^{2t} \cdot e^t f(t) = e^{3t} f(t)$

$\mathcal{L}^{-1}\left[\frac{e^{-5s}((s-2)^2-2(s-2)-15)}{((s-2)^2-2(s-2)+17)^2}\right] = e^{3(t-5)} f(t-5)\mu(t-5)$

h) $\mathcal{L}\left[e^{t-2} f(t-2)\mu(t-2)\right] = e^{-2s}\mathcal{L}\left[e^t f(t)\right] = e^{-2s} \cdot \frac{s^2-2s-15}{(s^2-2s+17)^2}$

9. $\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[((t-1)e^{t-1}\int_0^{t-1} u^8 e^{4u} du)\right]\mu(t-1)$

$= e^{-s}\mathcal{L}\left[te^t \int_0^t u^8 e^{4u} du\right] *$

$\mathcal{L}\left[t\int_0^t u^8 e^{4u} du\right] = -1 \cdot \frac{d}{ds}\left(\mathcal{L}\left[\int_0^t u^8 e^{4u} du\right]\right)$

$= -\frac{d}{ds}\left(\frac{1}{s} \cdot \mathcal{L}\left[t^8 e^{4t}\right]\right) **$

$\mathcal{L}\left[t^8\right] = \frac{8!}{s^9}$

$\mathcal{L}\left[t^8 e^{4t}\right] = \frac{8!}{(s-4)^9}$

volviendo a ** :

$\mathcal{L}\left[t\int_0^t u^8 e^{4u} du\right] = -\frac{d}{ds}\left(\frac{1}{s} \cdot \frac{8!}{(s-4)^9}\right)$

$= -\frac{d}{ds}\left(\frac{8!}{s(s-4)^9}\right)$, efectuando la derivada y simplificando se obtiene:

$\mathcal{L}\left[t\int_0^t u^8 e^{4u} du\right] = \frac{8!(10s-4)}{s^2(s-4)^{10}}$

$\mathcal{L}\left[te^t \int_0^t u^8 e^{4u} du\right] = \frac{8!(10(s-1)-4)}{(s-1)^2(s-1-4)^{10}} = \frac{8!(10s-14)}{(s-1)^2(s-5)^{10}}$

volviendo a *

$\mathcal{L}\left[f(t)\right] = e^{-s} \cdot \frac{8!(10s-14)}{(s-1)^2(s-5)^{10}}$

$= \frac{8! e^{-s}(10s-14)}{(s-1)^2(s-5)^{10}}$

10.

$L\frac{di}{dt} + Ri = E(t)$

$L\frac{di}{dt} + Ri = sen(t)(\mu(t) - \mu(t - \frac{3\pi}{2})$ $\qquad / \mathcal{L}$

$sI(s) - i(0^{+}) + 10I(s) = \mathcal{L}[sen(t)] - e^{-\frac{3\pi}{2}s}\mathcal{L}[sen(t + \frac{3\pi}{2})]$

$(s + 10)I(s) = \frac{1}{s^2+1} + \frac{e^{-\frac{3\pi}{2}s}s}{s^2+1}$

$I(s) = \frac{1}{(s+10)(s^2+1)} + \frac{e^{-\frac{3\pi}{2}s}s}{(s+10)(s^2+1)}$

$\frac{1}{(s+10)(s^2+1)} = \frac{A}{s+10} + \frac{Bs+C}{s^2+1}$

Los valores de A, B y C son:

$A = \frac{1}{101}$ $\qquad B = -\frac{1}{101}$ $\qquad C = \frac{10}{101}$

$\frac{s}{(s+10)(s^2+1)} = \frac{A'}{s+10} + \frac{B's+C'}{s^2+1}$

Los valores de A', B' y C' son:

$A' = -\frac{10}{101}$ $\qquad B' = \frac{10}{101}$ $\qquad C' = \frac{1}{101}$

$\mathcal{L}^{-1}[\frac{1}{(s+10)(s^2+1)}] = \frac{1}{101}e^{-10t} - \frac{1}{101}cos(t) + \frac{10}{101}sen(t)$

$\mathcal{L}^{-1}[\frac{s}{(s+10)(s^2+1)}] = -\frac{10}{101}e^{-10t} + \frac{10}{101}cos(t) + \frac{1}{101}sen(t)$

$i(t) =$
$(\frac{1}{101}e^{-10t} - \frac{1}{101}cos(t) + \frac{10}{101}sen(t)) +$
$(-\frac{10}{101}e^{-10(t-\frac{3\pi}{2})} + \frac{10}{101}cos(t - \frac{3\pi}{2}) + \frac{1}{101}sen(t - \frac{3\pi}{2}))\mu(t - \frac{3\pi}{2})$; $t > 0.$

11.

a)

$\mathcal{L}\left[\frac{f(t)}{e^{3t+1}}\right] = e^{-1}\mathcal{L}[e^{-3t}f(t)] = e^{-1}\frac{8(s+3)^2}{((s+3)^2+16)^2}$

b)

$\mathcal{L}[t\int_0^t f(u)du] = -\frac{d}{ds}\Big(\mathcal{L}[\int_0^t f(u)du]$

$= -\frac{d}{ds}\left(\frac{F(s)}{s}\right)$

$= -\frac{d}{ds}\left(\frac{8s}{(s^2+16)^2}\right)$

$= -\frac{8(16-3s^2)}{(s^2+16)^3}$

c)

$\int_0^{+\infty} e^{-3t}f(t)dt = F(3) = \frac{8\cdot 3^2}{(3^2+16)^2} = \frac{72}{625}$

d)

$\mathcal{L}^{-1}\left[\frac{e^{-2s}(s-2)^2}{(s^2-4s+20)^2}\right]$

$= \mathcal{L}^{-1}\left[\frac{e^{-2s}(s-2)^2}{((s-2)^2+16)^2}\right]$

$= \frac{1}{8}\mathcal{L}^{-1}\left[\frac{e^{-2s}8(s-2)^2}{((s-2)^2+16)^2}\right]$

$= \frac{1}{8}e^{2(t-2)}\big(sen(4(t-2) + 4(t-2)cos(4(t-2))\mu(t-2)$

12.

$\frac{dx}{dt} = 1 + 3x - y \qquad / \mathcal{L}$

$\frac{dy}{dt} = 2x \qquad / \mathcal{L}$

$sX(s) - x(0^+) = \frac{1}{s} + 3X(s) - Y(s)$

$sY(s) - y(0^+) = 2X(s)$

$(s-3)X(s) + Y(s) = \frac{1}{s} + 1 \qquad / \cdot s$

$2X(s) - sY(s) = -2$

$s(s-3)X(s)+sY(s)=s+1$ (1)

$2X(s)-sY(s)=-2$ (2)

(1)+(2) $s(s-3)X(s)+2X(s)=s-1$

$(s(s-3)+2)X(s)=s-1$

$(s^2-3s+2)X(s)=s-1$

$X(s)=\frac{s-1}{s^2-3s+2}=\frac{s-1}{(s-1)(s-2)}=\frac{1}{s-2}$

$x(t)=e^{2t}$

De $\frac{dx}{dt}=1+3x-y$ se obtiene: $y=1+3x-\frac{dx}{dt}$ (*)

Reemplazando $x(t)=e^{2t}$ en (*) se obtiene: $y=1+3e^{2t}-2e^{2t}=1+e^{2t}$

Luego

$x(t)=e^{2t}$

$y(t)=1+e^{2t}$

13.

$\frac{dx}{dt}+x+y=2cos(2t)$ $/\mathcal{L}$

$\frac{dy}{dt}+x-y=-2cos(2t)+2sen(2t)$ $/\mathcal{L}$

$sX(s)-x(0)+X(s)+Y(s)=\frac{2s}{s^2+4}$

$sY(s)-y(0)+X(s)-Y(s)=-\frac{2s}{s^2+4}+\frac{4}{s^2+4}$

$(s+1)X(s)+Y(s)=\frac{2s}{s^2+4}$ $/\cdot(-1)$

$X(s)+(s-1)Y(s)=\frac{-2s+4}{s^2+4}$ $/\cdot(s+1)$

$-(s+1)X(s)-Y(s)=-\frac{2s}{s^2+4}$ (1)

$(s+1)X(s)+(s^2-1)Y(s)=\frac{-2s^2-2s+4s+4}{s^2+4}$ (2)

(1)+(2) $(s^2-2)Y(s)=\frac{-2(s^2-2)}{s^2+4}$

$Y(s)=-\frac{2}{s^2+4}$

$y(t)=-sen(2t)$

Reemplazando $Y(s)=-\frac{2}{s^2+4}$ en $(s+1)X(s)+Y(s)=\frac{2s}{s^2+4}$ se obtiene:

$(s+1)X(s)-\frac{2}{s^2+4}=\frac{2s}{s^2+4}$

$X(s)=\frac{2}{s^2+4}$

$x(t)=sen(2t)$

Luego, $x(t)=sen(2t)$, $y(t)=-sen(2t)$

14.

$y''+y=\delta(t-2\pi)\ \ /\mathcal{L}$

$s^2Y(s)-y(0)s-y'(0)+Y(s)=\mathcal{L}[\delta(t-2\pi)]$

$s^2Y(s)-s+Y(s)=e^{-2\pi s}$

$(s^2+1)Y(s)=e^{-2\pi s}+s$

$Y(s)=\frac{e^{-2\pi s}}{s^2+1}+\frac{s}{s^2+1}$

$y(t)=sen(t-2\pi)\mu(t-2\pi)+cos(t)$

$y(t)=sen(t)\mu(t-2\pi)+cos(t)$

15.

$f(t)=sen(3t)+3tcos(3t)$

$\mathcal{L}[f(t)]=\frac{6s^2}{(s^2+9)^2}$

a) $\mathcal{L}\left[\frac{f(t)}{e^{2t+1}}\right] = \quad e^{-1}\mathcal{L}\left[e^{-2t}f(t)\right] = e^{-1} \cdot \frac{6(s+2)^2}{((s+2)^2+9)^2} = \quad \frac{6e^{-1}(s+2)^2}{((s+2)^2+9)^2}$

b) $\int_0^{+\infty} e^{-2t}f(t)dt = \frac{6 \cdot 2^2}{(2^2+9)^2} \quad = \frac{24}{169}$

c) $\mathcal{L}^{-1}\left[\frac{e^{-3s}(s-3)^2}{(s^2-6s+18)^2}\right] = \mathcal{L}^{-1}\left[\frac{e^{-3s}(s-3)^2}{((s-3)^2+9)^2}\right]$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+9)^2}\right] = \frac{1}{6}(sen(3t) + 3tcos(3t))$$

$$\mathcal{L}^{-1}\left[\frac{(s-3)^2}{((s-3)^2+9)^2}\right] = e^{3t} \cdot \frac{1}{6}(sen(3t) + 3tcos(3t)) = \frac{e^{3t}}{6}(sen(3t) + 3tcos(3t))$$

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}(s-3)^2}{((s-3)^2+9)^2}\right] = \frac{e^{3(t-3)}}{6}(sen(3(t-3)) + 3(t-3)cos(3(t-3)))\mu(t-3)$$

16.

$\mathcal{L}[f(t)] = \frac{2s^2-s-2}{s(s+2)(s-2)}$

a) $\mathcal{L}^{-1}[F(s)] = f(t)$

$\frac{2s^2-s-2}{s(s+2)(s-2)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-2}$ (descomposición en feacciones parciales)

Los valores de A, B y C son : $A = \frac{1}{2}$, $B = 1$, $C = \frac{1}{2}$

$f(t) = \mathcal{L}^{-1}\left[\frac{2s^2-s-2}{s(s+2)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{1/2}{s} + \frac{1}{s+2} + \frac{1/2}{s-2}\right]$

$f(t) = \frac{1}{2} + e^{-2t} + \frac{1}{2}e^{2t}$

b) $\mathcal{L}[e^{t-1}f(t-1)\mu(t-1)] \quad = e^{-s}\mathcal{L}[e^t f(t)]$

$$= e^{-s}\left(\frac{1/2}{s-1} + \frac{1}{s-1+2} + \frac{1/2}{s-1-2}\right)$$

$$= e^{-s}\left(\frac{1}{2(s-1)} + \frac{1}{s+1} + \frac{1}{2(s-3)}\right)$$

17.

$y'' - y' = e^t \quad / \mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = \frac{1}{s-1}$$

$$s^2Y(s) - 2s - 2 - (sY(s) - 2) = \frac{1}{s-1}$$

$$(s^2 - s)Y(s) = 2s + \frac{1}{s-1}$$

$$s(s-1)Y(s) = 2s + \frac{1}{s-1}$$

$$Y(s) = \frac{2}{s-1} + \frac{1}{s(s-1)^2}$$

$$\frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Los valores de $A,\ B$ y C son : $A = 1\ ,\ B = \ -1\ ,\ C = 1$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s-1} + \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}\right]$$

$$y(t) = e^t + 1 + te^t$$

18.

Sea $y(0) = a\ ,\ y'(0) = b$ y $\mathcal{L}[y(t)] = u$

$$-\frac{d}{ds}(s^2u - as - b) + su - a + \frac{d}{ds}(su - a) + u = 0$$

$$-2su - s^2u' + a + su - a + u + su' + u = 0$$

$$-(s^2 - s)u' + (2 - s)u = 0$$

$$u' - \frac{2-s}{s(s-1)}u = 0$$

$$\therefore \qquad u = ce^{\int \frac{2-s}{s(s-1)}ds} = ce^{\int \left(\frac{1}{s-1} - \frac{2}{s}\right)ds} = ce^{ln(s-1) - 2ln(s)} = \frac{c(s-1)}{s^2}$$

$$u = c\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

$$\therefore \qquad y(t) = c(1 - t) \qquad c \neq 0$$

19.

$$\mathcal{L}[g(t)] = \frac{1}{s}\ \mathcal{L}[e^t(t - [\![t]\!])] - \frac{1}{s}\int_0^1 xe^x dx$$

$f(t) = t - [\![t]\!]$ tiene periodo $p = 1$

$$\therefore \qquad \mathcal{L}[t - [\![t]\!]] = \frac{1}{1-e^{-s}}\int_0^1 te^{-st}dt = \frac{1}{1-e^{-s}}\left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2} - \frac{1}{s(e^s-1)}$$

$$\mathcal{L}[e^t(t - [\![t]\!])] = \frac{1}{(s-1)^2} - \frac{1}{(s-1)(e^{s-1}-1)}$$

Además $\int_0^1 xe^x dx = (xe^x - e^x)\Big|_0^1 = 1$

$$\therefore \qquad \mathcal{L}[g(t)] = \frac{1}{s(s-1)^2} - \frac{1}{s(s-1)(e^{s-1}-1)} - \frac{1}{s}$$

20.

a) $f(t) = -\mu(t-\pi)sen(t) = \mu(t-\pi)sen(t-\pi)$

$g(t) = (1-\mu(t-\pi))t = t - \mu(t-\pi)(t-\pi) - \pi\mu(t-\pi)$

b) $\mathcal{L}[(f*g)(t)] = \frac{e^{-\pi s}}{s^2+1}\left(\frac{1}{s^2} - \frac{e^{-\pi s}}{s^2} - \frac{\pi e^{-\pi s}}{s}\right)$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] = t - sen(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] = 1 - cos(t)$$

$\therefore$

$(f*g)(t)$

$= \mu(t-\pi)(t-\pi-sen(t-\pi) - \mu(t-2\pi)(t-2\pi-sen(t-2\pi) - \pi\mu(t-2\pi)(1-cos(t-2\pi))$

$= \mu(t-\pi)(t-\pi+sen(t) - \mu(t-2\pi)(t-\pi-sen(t) - \pi cos(t))$